

# SQUARE NUMBERS, SPANNING TREES AND INVARIANTS OF ACHIRAL KNOTS

*This is a preprint. I would be grateful for any comments and corrections!*

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**Abstract.** We give constructions to realize an odd number, which is representable as sum of two squares, as determinant of an achiral knot, thus proving that these are exactly the numbers occurring as such determinants. Later we study which numbers occur as determinants of prime alternating achiral knots, and obtain a complete result for perfect squares. Using the checkerboard coloring, then an application is given to the number of spanning trees in planar self-dual graphs. Another application are some enumeration results on achiral rational knots. Finally, we describe the leading coefficients of the Alexander and skein polynomial of alternating achiral knots.

**Keywords:** alternating knots, homogeneous knots, achiral knots, Alexander polynomial, HOMFLY polynomial, determinant, spanning tree.

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## 1. Introduction

The main problem of knot theory is to distinguish knots (or links), i.e., smooth embeddings of  $S^1$  (or several copies of it) into  $\mathbb{R}^3$  or  $S^3$  up to isotopy. A main tool for this is to find *invariants* of knots, i.e., maps of knot diagrams into some algebraic structure, which are invariant under Reidemeister's moves. A family of most popular such invariants are the polynomial invariants, associating to each knot an element in some one- or two-variable (Laurent) polynomial ring over  $\mathbb{Z}$ . Given a knot invariant, beside distinguishing knots with it, one is also interested which properties of knots it reflects, and in which way.

One of the most intuitive ways to associate to a knot (or link) another one is to consider its *obverse*, or mirror image, obtained by reversing the orientation of the ambient space. The knot (or link) is called *achiral* (or synonymously *amphicheiral*), if it coincides (up to isotopy) with its mirror image, and *chiral* otherwise. When considering orientation of the *knot*, then we distinguish among achiral knots between +achiral and -achiral ones, dependingly on whether the deformation into the mirror image preserves or reverses the orientation of the knot. (For links one has to attach a sign to each component, i.e. embedded circle, and take into account possible permutations of the components.)

When the *Jones polynomial*  $V$  [J] appeared in 1984, one of its (at that time) spectacular features was that it was (in general) able to distinguish between a knot and its obverse by virtue of having distinct values on both, and (hence) so were its generalizations, the *HOMFLY, or skein, polynomial*  $P$  [F&] and the *Kauffman polynomial*  $F$  [Ka]. The  $V$ ,  $P$  and  $F$  polynomials of achiral knots have the special property to be *self-conjugate*, that is, invariant when one of the variables is replaced by its inverse. Their decades-old predecessor, the *Alexander polynomial*  $\Delta$  [Al], a knot invariant with values in  $\mathbb{Z}[t, t^{-1}]$ , was known always to take the same value on a knot and its mirror image. Nevertheless, contrarily to the common belief,  $\Delta$  can also be used to detect chirality (the property of a knot to be distinct from its mirror image) by considering its value  $\Delta(-1)$ , called *determinant*.

The aim of this paper is to study invariants of achiral knots and to relate some properties of their determinants to the classical topic in number theory of representations of integers as sums of two squares.

In §2 we begin with recalling a criterion for the Alexander polynomial of an achiral knot via the determinant, which follows from Murasugi's work on the signature and the Lickorish-Millett value of the Jones polynomial. This conditions show that, paradoxly formulated, although the Alexander polynomial cannot distinguish between a knot and its mirror image, it can still sometimes show that they are distinct.

After collecting some number theoretic preliminaries in §3, we show then in §4.1 that the condition of §2 is in fact a reduction modulo 36 of the exact arithmetic description of numbers, occurring as determinants of achiral knots. Namely, an odd natural number is the determinant of an achiral knot if and only if it is the sum of two squares. The ‘only if’ part of this statement was an observation of Hartley and Kawauchi in [HK]. Our aim will be to show the ‘if’ part, that is, given a sum of two squares, to realize it as the determinant of an achiral knot (theorem 4.1). The main tool used is the definition of the determinant by means of Kauffman's state model for the Jones polynomial [Ka2].

Then we attempt to refine our construction, by producing achiral knots (of given determinant) with additional properties: prime and/or alternating. Although it turns out, that one of these properties can always easily be achieved, the situation reveals much harder when demanding them both altogether. We investigate this problem in §4.2. Now, the correspondence of §4.1 does not hold completely, and there are exceptional values of the determinant, that cannot be realized. To show that 9 and 49 are such, we prove a quadratic improvement of Crowell's (lower) bound for the determinant of an alternating knot in terms of its crossing number [C2], in the case the knot is achiral (proposition 4.2). We obtain then a complete result about which perfect squares can be realized as determinants of prime alternating achiral knots (theorem 4.4).

In §4.3 we consider the problem to describe determinants of unknotting number one achiral knots. In this case the description is even less clear, as we show by several examples.

Then we give some applications, including enumeration results on rational knots in §5, and a translation of the previously established properties to the number of spanning trees in planar self-dual graphs in §6.

Subsequently, in §7 we prove some further (at parts still remaining conjectural) properties of the Alexander and skein polynomial of at least large classes of achiral knots, which would allow to decide about chirality (the lack of an isotopy to the mirror image) in a yet different way, at least for these knot classes. These properties concern the leading coefficients of the polynomials, and are closely related to Murasugi's  $*$ -product. We prove in particular that perfect squares are exactly the numbers occurring as leading coefficients of the Alexander polynomial of alternating achiral knots, thus improving the previously known necessary condition of non-primeness (corollary 7.1). These results have been obtained with the same arguments independently (but somewhat later) by C. Weber and Q. H. Cám Ván [VW].

Several open problems are suggested during the discussion throughout the paper. These problems appear to be involved enough already for knots, so that we waived on an analogous study of links (which are the cases covering the even natural numbers). For links, also the unpleasant issue of component orientations becomes relevant.

## 2. Detecting chirality with the Alexander polynomial

In the following we will be concerned with the value  $\Delta(-1)$  of the Alexander polynomial, where  $\Delta$  is normalized so that  $\Delta(t) = \Delta(1/t)$  and  $\Delta(1) = 1$ . Up to sign, this numerical invariant can be interpreted as the order of the homology group (over  $\mathbb{Z}$ ) of the (double) branched covering of  $S^3$  over  $K$  associated to the canonical homomorphism  $\pi_1(S^3 \setminus K) \rightarrow \mathbb{Z}_2$  and carries the name “*determinant*” because of its expression (up to sign) as the determinant of a Seifert [Ro, p. 213] or Goeritz [G] matrix.

To introduce some mathematical notations of the objects thus occurring, let  $D_K$  be the double branched cover of  $S^3$  over a knot  $K$ , and let  $H_1(D_K)$  be its homology group (over  $\mathbb{Z}$ ). We write then  $\det(K) = |\Delta_K(-1)| = |H_1(D_K)|$ .

We start first by description of two special cases of the exact property of the determinant of achiral knots, which we will formulate subsequently, because they have occurred in independent contexts and deserve mention in their own right. They allow to decide about chirality of a knot  $K$ , at least for  $11/18$  of the possible values of  $\Delta_K(-1)$ .

There is an observation (originally likely, at least implicitly, due to Murasugi [Mu], and applied explicitly in [St]), using the sign of the value  $\Delta(-1)$  (with  $\Delta$  normalized as said). The information of this sign is equivalent to the residue  $\sigma \bmod 4$ , where  $\sigma$  denotes the *signature*. Whenever  $\Delta(-1) < 0$ , we have  $\sigma \equiv 2 \bmod 4$ , so in particular  $\sigma \neq 0$ , and the knot cannot be achiral. This argument works e.g. for the knot  $9_{42}$  in the tables of [Ro, appendix], which became famous by sharing the same  $V$ ,  $P$  and  $F$  polynomial with its obverse, since its polynomials are all self-conjugate.

Another way to deduce chirality from the determinant is to use the sign of the Lickorish-Millett value  $V(e^{\pi i/3})$  [LM2]. Attention to it was drawn in [Tr], where it was used to calculate unknotting numbers. Using some of the ideas there, in [St3] we observed that this sign implies that if for an achiral knot  $3 \mid \Delta(-1)$ , then already  $9 \mid \Delta(-1)$ . Thus for example also the chirality of  $7_7$  can be seen already from its Alexander polynomial, as in this case  $\Delta(-1) = 21$  (although the Murasugi trick does not work here, and indeed  $\sigma = 0$ ).

Combining both criteria, we arrive in summary to

**Proposition 2.1** For any achiral knot  $K$  we have  $|\Delta_K(-1)| \bmod 36 \in \{1, 5, 9, 13, 17, 25, 29\}$ . □

An easy verification shows that all these residues indeed occur.

In view of these opportunities to extract chirality information out of  $\Delta$ , it appears appropriate to introduce a clear distinction between the terms ‘detecting chirality of  $K$ ’, meant in the sense ‘showing that  $K$  and  $!K$  are not the same knot’ (which can be achieved by the above tricks) and ‘distinguishing between  $K$  and  $!K$ ’, meant in the sense ‘identifying for a given diagram, known *a priori* to belong to either  $K$  or  $!K$ , to which one of both it belongs’ (what they cannot accomplish, but what is the usually imagined situation where some of the other polynomials is not self-conjugate).

Here is a small arithmetic consequence. It is elementary, but is included because of its knot theoretical interpretation and as it is the starting point of exhibiting some more interesting phenomena described in the next sections.

Recall, that a knot  $K$  is *rational* (or 2-bridge), if it has an embedding with a Morse function having only four critical points (2 maxima and 2 minima). Such knots were classified by Schubert [Sh], and can be alternatively described by their Conway notation [Co]. See for example [Ad, §2.3] for a detailed description. It is well-known that rational knots are alternating (see [BZ, proposition 12.14, p. 189]).

**Corollary 2.1** Let  $p/q$  for  $(p,q) = 1$ ,  $p$  odd be expressible as the continued fraction

$$[[a_1, \dots, a_n, a_n, \dots, a_1]] := a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots a_n + \cfrac{1}{a_1}}}$$

for a palindromic sequence  $(a_1, \dots, a_n, a_n, \dots, a_1)$  of even length (with the usual conventions  $\frac{1}{0} = \infty$ ,  $\infty + * = \infty$ ,  $\frac{1}{\infty} = 0$  for the degenerate cases). Then  $|p| \equiv 1$  or  $5 \pmod{12}$ .

**Proof.** Observe that  $|p|$  is the determinant of the achiral rational knot with Conway notation  $(a_1 \dots a_n a_n \dots a_1)$ . The above proposition 2.1 leaves us only with explaining why  $9 \nmid p$ . The implication  $3 \mid \det(K) \implies 9 \mid \det(K)$  for  $K$  achiral using  $V(e^{\pi i/3})$  follows from the fact that the number of torsion coefficients divisible by 3 of the  $\mathbb{Z}$ -module  $H_1(D_K)$ , counted by  $V(e^{\pi i/3})$ , is even. However, for a rational knot  $K$ ,  $H_1(D_K)$  is cyclic and non-trivial ( $D_K$  is a lens space), so that there is only one torsion number at all. Thus  $H_1(D_K)$  for any achiral rational knot  $K$  cannot have any 3-torsion.  $\square$

### 3. Number theoretic preliminaries

According to a claim of Fermat, written about 1640 on the margins of his copy of Euclid's "Elements", proved in 1754 by Euler, and further simplified to the length of "one sentence" in [Z3], any prime of the form  $4x+1$  can be written as the sum of two squares. More generally, any natural number  $n$  is the sum of two squares if and only if any prime of the form  $4x+3$  occurs in the prime decomposition of  $n$  with an even power, and it is the sum of the squares of two coprime numbers if and only if such primes do not occur at all in the prime decomposition of  $n$ .

The number of representations as the sum of two squares is given by the formula

$$r_2(n) := \frac{1}{4} \#\{(m_1, m_2) \in \mathbb{Z}^2 : m_1^2 + m_2^2 = n\} = \sum_{d|n} \left( \frac{-4}{d} \right) = \#\{x \in \mathbb{N} : 4x+1 \mid n\} - \#\{x \in \mathbb{N} : 4x+3 \mid n\}, \quad (1)$$

which has also an interpretation in the theory of modular forms (see [HW, (16.9.2) and theorem 278, p. 275] and [Z2]).

A number theoretic explanation of (1) is as follows: If we denote by  $\chi$  the (primitive) character  $(\frac{-4}{\cdot})$ , with  $-4$  being the discriminant of the field of the Gauß numbers  $\mathbb{Q}[i]$ , we have for  $\Re(s) > 1$ , using that  $\mathbb{Q}[i]$  has class number 1 and 4 units, that

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = \zeta_{\mathbb{Q}[i]}(s) = \zeta(s)L(s, \chi) = \prod_{p \text{ prime}} \frac{1}{(1-p^{-s})(1-\chi(p)p^{-s})}, \quad (2)$$

from which the formula follows by considering the Taylor expansion in  $p^{-s}$  of the different factors in the product. (This series converges for  $\Re(s) > 1$ .)

The  $\zeta$ -function identities also give a formula for

$$r_2^0(n) := \frac{1}{4} \#\{(a, b) \in \mathbb{Z}^2 : (a, b) = 1, a^2 + b^2 = n\}, \quad (3)$$

the number of representations of  $n$  as the sum of squares of coprime numbers.

We have

$$\sum_{n=1}^{\infty} \frac{r_2^0(n)}{n^s} = \frac{\zeta_{\mathbb{Q}[i]}(s)}{\zeta(2s)} = \frac{\zeta(s)L(s, \chi)}{\zeta(2s)} = \prod_{p \text{ prime}} \frac{1-p^{-2s}}{(1-p^{-s})(1-\chi(p)p^{-s})} = (1+2^{-s}) \prod_{\substack{p \equiv 1(4) \\ \text{prime}}} \frac{1+p^{-s}}{1-p^{-s}}. \quad (4)$$

Thus

$$r_2^0(n) = \begin{cases} 2^k & \text{if } n = p_1^{k_1} \cdot \dots \cdot p_k^{k_k} \text{ or } 2p_1^{k_1} \cdot \dots \cdot p_k^{k_k} \text{ with } p_1 < p_2 < \dots < p_k \text{ primes } \equiv 1 \pmod{4} \text{ and } k_i > 0 \\ 0 & \text{else} \end{cases}. \quad (5)$$

Note, that for  $n > 1$ ,

$$r_2^0(n) = \#\{(a, b) \in \mathbb{N}^2 : (a, b) = 1, a^2 + b^2 = n\}, \quad (6)$$

and for  $n > 2$  we have

$$\frac{r_2^0(n)}{2} = \#\{(a, b) \in \mathbb{N}^2 : (a, b) = 1, a \leq b, a^2 + b^2 = n\}. \quad (7)$$

## 4. Sums of two squares and determinants of achiral knots

### 4.1. Realizing sums of two squares as determinants

The aim of this section is to establish a, partially conjectural, correspondence between sums of two squares and the determinant of achiral links. The study of this relation was first initiated in [HK], where it was observed that a result of Goeritz [G] implies that the determinant of an achiral knot is the sum of two squares. We shall here show the converse. In fact, we have

**Theorem 4.1** An odd natural number  $n$  occurs as determinant of an achiral knot  $K$  if and only if  $n$  is the sum of two squares  $a^2 + b^2$ . More specifically,  $K$  can be chosen to be alternating or prime, and if one can choose  $a$  and  $b$  to be coprime, then we can even take  $K$  to be rational (or 2-bridge).

Note, that we have given another argument from that of Hartley and Kawauchi for the reverse implication “modulo 36”: it was observed above how the signature and the Lickorish-Millett value of the Jones polynomial imply that if  $n$  is the determinant of an achiral knot, then  $n \bmod 36 \in \{1, 5, 9, 13, 17, 25, 29\}$ . These are exactly the congruences which odd sums of two squares leave modulo 36. Clearly, not every number satisfying these congruences is the sum of two squares. The simplest example is 77. And indeed, this number does not occur as determinant of any achiral knot of  $\leq 16$  crossings.

Additionally to the general case, we also have a complete statement for rational knots. (An analogue for arborescent knots seems possible by applying the classification result of Bonahon and Siebenmann [BS].)

**Theorem 4.2** An odd natural number  $n$  is the determinant of an achiral rational knot if and only if it is the sum of the squares of two coprime numbers.

The coprimality condition is clearly restrictive – for example 49 and 121 are not sums of the squares of two coprime numbers. Moreover, it also implies the congruence modulo 12 proved in corollary 2.1.

Fermat’s theorem can be now knot-theoretically reformulated for example as

**Corollary 4.1** If  $n = 4x + 1$  is a prime, then there is a rational achiral knot with determinant  $n$ .

**Proof.** We have  $n = a^2 + b^2$  and as  $n$  is prime,  $a$  and  $b$  must be coprime.  $\square$

We start by a proof of theorem 4.2. For this recall Krebes’s invariant defined in [Kr]. Any tangle  $T$  can be expressed by its coefficients in the the Kauffman bracket skein module of the room with four in/outputs (see [St6]):

$$\boxed{T} = A \boxed{|} + B \boxed{\circlearrowleft}. \quad (8)$$

**Definition 4.1** For a tangle  , we call  the *numerator closure* of  $T$  and  the *denominator closure* of  $T$ .

When specializing the bracket variable to  $\sqrt{i}$  ( $i$  denotes henceforth  $\sqrt{-1}$ ),  $A$  and  $B$  in (8) become scalars.

Then Krebes’s invariant can be defined by

$$\text{Kr}(T) := \frac{A}{B} = (A, B) \in \mathbb{Z} \times \mathbb{Z}/(p, q) \sim (-p, -q).$$

The denominator and numerator of this “fraction” give the determinants of the two closures of  $T$ .

**Proof of theorem 4.2.** A rational achiral knot  $(a_1 \dots a_n a_n \dots a_1)$  is of the form

$$K = \text{Diagram} , \quad (9)$$

where  $T = (a_1 \dots a_n)$  is a rational tangle and  $\bar{T}$  its mirror image. Because of connectivity reasons  $T$  must be of homotopy type  $\text{Diagram}$  (or  $\text{Diagram}$ , i.e.  $\text{Kr}(T) = [[a_1, \dots, a_n]] = \frac{p}{q}$  with  $(p, q) = 1$  and exactly one of  $p$  and  $q$  is odd. Thus  $K$  is the numerator closure of the tangle sum  $T + \bar{T}^T$ , where  $.^T$  denotes transposition. By the calculus introduced by Krebes, his invariant is additive under tangle sum, and invertive under transposition, and so we have

$$\frac{\det(K)}{*} = \text{Kr}(T + \bar{T}^T) = \text{Kr}(T) + \frac{1}{\text{Kr}(T)} = \frac{p}{q} + \frac{q}{p} = \frac{p^2 + q^2}{pq}.$$

To justify our choice of sign in this calculation, that is, that the determinant is  $p^2 + q^2$  rather than  $p^2 - q^2$ , it suffices to keep in mind that the diagram (9) is alternating and in calculating the bracket of alternating diagrams no cancellations occur, as explained also in [Kr]. Thus we have the ‘only if’ part.

For the ‘if’ part note that if  $a$  and  $b$  are coprime, then  $\frac{a}{b}$  can be expressed by an continued fraction, and hence as  $\text{Kr}(T)$  for some rational tangle  $T$ . Then  $a^2 + b^2$  (with the above remark on signs) is the determinant of the achiral knot shown in (9).  $\square$

Now we modify the second part of the proof to deduce theorem 4.1. In the following we use Conway’s notation for tangle sum and product. (See for example again [Ad, §2.3] for a detailed description.)

**Proof of theorem 4.1.** Let  $n = p^2 + q^2$ . If  $q = 0$  then  $K = T(2, p)\#T(2, -p)$  ( $T(2, p)$  denoting the  $(2, p)$ -torus knot) is an easy example, so let  $q \neq 0$ . Krebes shows that for any pair  $(p, q)$  with at least one of  $p$  and  $q$  odd there is a(n arborescent) tangle  $T$  with  $\text{Kr}(T) = \frac{p}{q}$ . In fact,  $T$  can be chosen to be the connected sum of a rational tangle and a knot of the type  $T(2, p)$  (which can be done in a way the tangle remains alternating). Then again consider the knot in (9) (it is a knot because of the parities of  $p$  and  $q$ ), and from the proof of theorem 4.2 one sees that it has the desired determinant  $n$ .

The knots constructed in (9) then are all alternating. It remains to show that they can be made prime (possibly sacrificing alternation). If  $(p, q) = 1$ , then  $K$  is rational, and hence prime. Thus let  $n = (p, q) > 1$ . Then we can choose  $T$  to be

$$T = \text{Diagram} , \quad (10)$$

that is, in Conway’s notation  $T = T' \cdot (0, n)$  with a tangle  $T'$  being a rational tangle  $a'/b'$  with  $a' = p/n, b' = q/n$ .

Now replace the 0-tangle in (10) by the (flipped) “KT-grabber” tangle  $KT$  in [Bl, Fl]. By the same argument as in [Fl] (or see also [KL, Va]), the tangle  $T_1 = T' \cdot (KT \cdot 0, n)$  becomes prime, and hence so is then the knot  $K_1 = \bar{T}_1 \cdot \bar{T}_1$  by proposition 1.3 of [Bl] (bar denotes tangle closure). As in [Bl],  $K_1$  and  $K$  have the same Alexander polynomial, so in particular the same determinant.  $\square$

**Example 4.1** To demonstrate the elegance of theorem 4.1 as a chirality criterion, we remark that among the prime knots of  $\leq 10$  crossings (denoted henceforth according to Rolfsen’s tables [Ro, appendix]) there are 6 chiral knots with self-conjugate HOMFLY polynomial  $-9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}$  and  $10_{125}$ , and this method shows chirality of four of them  $-9_{42}, 10_{71}, 10_{104}$  and  $10_{125}$ , including the two examples ( $9_{42}$  and  $10_{71}$ ) where additionally even the Kauffman polynomial is self-conjugate. (For  $9_{42}$  and  $10_{125}$  the congruence modulo 4 is violated, so that, as remarked on several other places, the signature works as well.)

**Remark 4.1** Since slice knots have square determinant, it also follows that if there exists a rational knot  $S(p, q)$  which is at the same time achiral and slice, then it will correspond to a Pythagorean triple, that is, be of the Schubert form  $S((m^2 + n^2)^2, 2mn(m^2 - n^2))$  with  $m$  and  $n$  coprime.

With regard to theorem 4.1, we conjecture an analogous statement to hold for links.

**Conjecture 4.1** An even natural number  $n$  occurs as determinant of an achiral link if and only if  $n$  is the sum of two squares.

As a remark on links, note that by the above description of numbers which are sums of two squares, this set is closed under multiplication, corresponding on the level of determinants of links to taking connected sums. Thus it would suffice to prove conjecture 4.1 just for prime links.

More number theoretic results on the square representations (which by the said above can also be transcribed knot-theoretically) may be found in [K, W].

## 4.2. Determinants of prime alternating achiral knots

Theorem 4.1 naturally suggests the question in how far the properties alternation and primeness can be combined when realizing a sum of two squares as determinant of achiral knots.

In this case, the situation is much more difficult, though. It is easy to see that not every (odd) sum of 2 squares can be realized. The first (and trivial) example is 1, since the only alternating knot with such determinant is the unknot [C2], and it is by definition not prime. However, there are further examples.

**Proposition 4.1** Let  $n = 9$  or  $n = 49$ . Then there is no prime alternating achiral knot of determinant  $n$ .

By the above cited result of Crowell, one has a bound on the crossing number of an alternating knot of given determinant, so could check for any  $n$  in finite time whether it is realized or not. This renders the check for  $n = 9$  easy. However, the estimate we obtain from Crowell's inequality is intractable in any practical sense for  $n = 49$ . We give an improvement of Crowell's result for achiral alternating links, which, although not completely sharp, is enough for our purpose.

For the understanding and the proof of this result we recall some standard terminology for knot diagrams.

**Definition 4.2** The diagram on the left of figure 1 is called *connected sum*  $A \# B$  of the diagrams  $A$  and  $B$ . If a diagram  $D$  can be represented as the connected sum of diagrams  $A$  and  $B$ , such that both  $A$  and  $B$  have at least one crossing, then  $D$  is called *disconnected* (or composite), else it is called *connected* (or prime). Equivalently, a diagram is prime if any closed curve intersecting it in exactly two points, does not contain a crossing in one of its in- or exterior.

If a diagram  $D$  can be written as  $D_1 \# D_2 \# \dots \# D_n$ , and all  $D_i$  are prime, then they are called the *prime (or connected) components/factors* of  $D$ .

**Definition 4.3** The diagram is *split*, if there is a closed curve not intersecting it, but which contains parts of the diagram in both its in- and exterior.

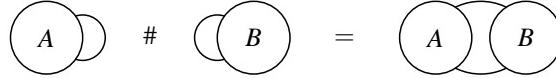


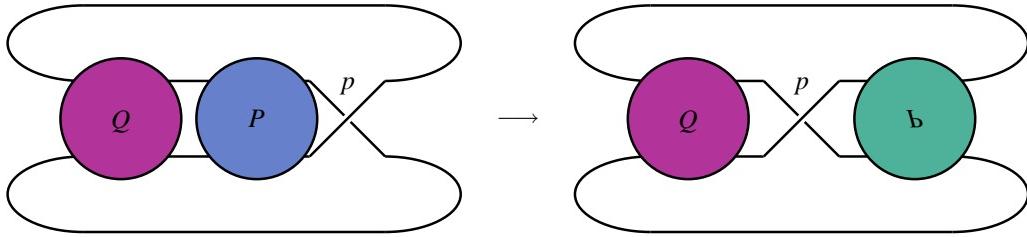
Figure 1

By [Me] an alternating link is prime/split iff any alternating diagram of it is so.

**Definition 4.4** A crossing  $q$  in a link diagram  $D$  is called *nugatory*, if there is a closed (smooth) plane curve  $\gamma$  intersecting  $D$  transversely in  $q$  and nowhere else. A diagram is called *reduced* if it has no nugatory crossings.

By [Ka2, Mu3, Th], each alternating reduced diagram is of minimal crossing number (for the link it represents).

**Definition 4.5** A *flype* is a move on a diagram shown in figure 4.



**Figure 2:** A flype near the crossing  $p$

By the fundamental work of [MT], for two alternating diagrams of the same alternating link, there is a sequence of flypes (and  $S^2$ -moves) taking the one diagram into the other.

**Definition 4.6** To define the *sign* of a crossing in a link diagram, choose an orientation of the link. The sign is then given as follows:

$$\begin{array}{c} \nearrow \searrow \\ \times \end{array} \quad \begin{array}{c} \nearrow \nearrow \\ \times \end{array} . \quad (11)$$

A crossing of sign  $+$  we call *positive*, and a crossing of sign  $-$  we call *negative*. Note, that the definition requires a link orientation, but for a *knot* it is independent on which of its both possible choices is taken.

The *writhe* is the sum of the signs of all crossings in a diagram. It is invariant under simultaneous reversal of orientation of *all* components of the diagram, so is in particular well-defined for unoriented *knot* diagrams. It may, however, change if some (but not all) components of a link diagram are reverted.

Crowell's result about the determinant of alternating links is

**Theorem 4.3** (Crowell) If  $L$  is a non-split alternating link of  $n$  crossings, then  $\det(L) \geq n$ , and if  $L$  is not the  $(2, n)$ -torus link, then  $\det(L) \geq 2n - 3$ .

We will show

**Proposition 4.2** If  $L$  is an alternating non-split achiral link of  $2n$  crossings, then  $\det(L) \geq n(n - 3)$ .

Since we use the checkerboard colorings for the proof, our result holds for the most general notion of achirality for links – we allow the isotopy taking a link to its mirror image to interchange components and/or preserve or reverse their orientations in an arbitrary way. We will define, however, checkerboard colorings later, in §6, so that we defer the proof of proposition 4.2 to that later stage.

**Remark 4.2** The condition the crossing number of  $L$  to be even is no restriction. The crossing number of any alternating achiral (in the most general sense, as remarked after Proposition 4.2) link diagram is even by [MT], since flypes preserve the writhe of the alternating diagram, and reversal of any single component alters the writhe by a multiple of 4 (any two components have an even number of common crossings by the Jordan curve theorem). Thus the writhe must be even, and hence so must be the crossing number.

**Proof of proposition 4.1.** This is now feasible. Check all the alternating achiral knots in the tables of [HT] up to 16 crossings.  $\square$

There is some possibility that the values of proposition 4.1 are indeed the only exceptions.

**Conjecture 4.2** Let  $n$  be an odd natural number. Then  $n$  is the determinant of a prime alternating achiral knot if and only if  $n$  is the sum of two squares and  $n \notin \{1, 9, 49\}$ .

At least this is true up to  $n \leq 2000$ . A full confirmation of this conjecture is so far not possible, but we obtain a complete statement for  $n$  being a square.

**Theorem 4.4** Let  $n$  be an odd square. Then  $n$  is the determinant of a prime alternating achiral knot if and only if  $n \notin \{1, 9, 49\}$ .

Recall, that a knot  $K$  is called *strongly achiral*, if it admits an embedding into  $S^3$  pointwise fixed by the (orientation-reversing) involution  $(x, y, z) \mapsto (-x, -y, -z)$ . Again dependingly on the effect of this involution on the orientation of the knot we distinguish between strongly +achiral and strongly -achiral knots.

Let  $\mathcal{L} = \mathbb{Q}[t, t^{-1}]$  be the Laurent polynomial ring in one variable. For  $F, G \in \mathcal{L}$  write  $F \sim G$  if  $F$  and  $G$  differ by a multiplicative unit in  $\mathbb{Z}[t, t^{-1}]$ , that is,  $F(t) = \pm t^n G(t)$  for some  $n \in \mathbb{Z}$ .

The result of [HK] is the following.

**Theorem 4.5** ([HK]) If  $K$  is strongly negative amphicheiral, then  $\Delta(t^2) \sim F(t)F(-t)$  for some  $F \in \mathcal{L}$  with  $F(-t) \sim F(t^{-1})$ . If  $K$  is strongly positive amphicheiral, then  $\Delta(t) = F(t)^2$  for some  $F \in \mathcal{L}$  with  $F(t) \sim F(t^{-1})$ .

This theorem will not be used here, but provides some heuristics for the proof of theorem 4.4, and will come to more detailed mention later, so it is possibly appropriate to introduce it here.

**Proof of theorem 4.4.** Since  $n$  is a square, it is suggestive by the result of [HK] to consider strongly +achiral knots as candidates to realize  $n$  as determinant. We consider diagrams of the type

$$D(T_1) = \begin{array}{c} \text{Diagram of } T_1 \text{ consisting of a vertical line with a horizontal loop attached, labeled } i\Gamma \text{ above the loop and } T_1 \text{ below the line.} \end{array}$$

Define a pairing  $\langle T_1, T_2 \rangle$  on the diagram algebra  $DS_3(A = \sqrt{i})$  (see [Ka]) by table 1 (compare also to the pairing  $\langle \cdot, \cdot \rangle_3$  in §4 of [St5]).

Then  $\det(D(T_1)) = \langle T_1, T_1 \rangle$ . Let  $T$  be a tangle

$$\begin{array}{c} \text{Diagram of } T \text{ showing a vertical line with a crossing, labeled } T_1 \text{ above the crossing and } T_2 \text{ below the line.} \end{array}$$

with  $T_1 = A \mid \mid + B \curvearrowleft$  and  $T_2 = X \mid \mid + Y \curvearrowleft$ . Then

$$T = a \mid \mid \mid + b \mid \curvearrowleft + c \curvearrowleft \mid + d \curvearrowright \curvearrowleft + e \curvearrowright \curvearrowleft$$

$T_1 \backslash T_2$			$\diagup$	$\diagdown$	$\diagup\diagdown$	$\diagdown\diagup$
	0	0	0	1	1	
	0	1	0	0	0	
$\diagup$	0	0	1	0	0	
$\diagdown$	1	0	0	0	1	
$\diagup\diagdown$	1	0	0	1	0	

**Table 1**

with

$$\begin{aligned} a &= XA \\ b = d &= BX \\ c &= AX + BY + AY \\ e &= BY, \end{aligned}$$

and

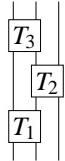
$$\langle T, T \rangle = [(X+Y)(A+B)]^2.$$

Whenever  $(X, Y)$  and  $(A, B)$  are relatively prime, and  $X, Y, A, B > 0$ , one can substitute rational tangles for  $T_{1,2}$  obtaining a prime alternating diagram of a strongly +achiral knot. Setting  $X = B = 1$  and varying  $Y$  and  $A$ , we see that we can cover all cases when  $n = p^2$  with  $p$  composite.

Since we dealt with  $p = 1$ , it remains to consider  $p$  prime. If  $p \equiv 1 \pmod{4}$ , then (1) shows that  $n$  has a non-trivial representation as sum of two squares, which then must be coprime. In this case there is an achiral rational knot of determinant  $n$ .

Thus assume  $n = p^2$  with  $p \equiv 3 \pmod{4}$  prime. We show now that almost all (not necessarily prime)  $p \equiv 3 \pmod{4}$  can be realized.

Consider diagrams  $D(T)$  for  $T$  of the form



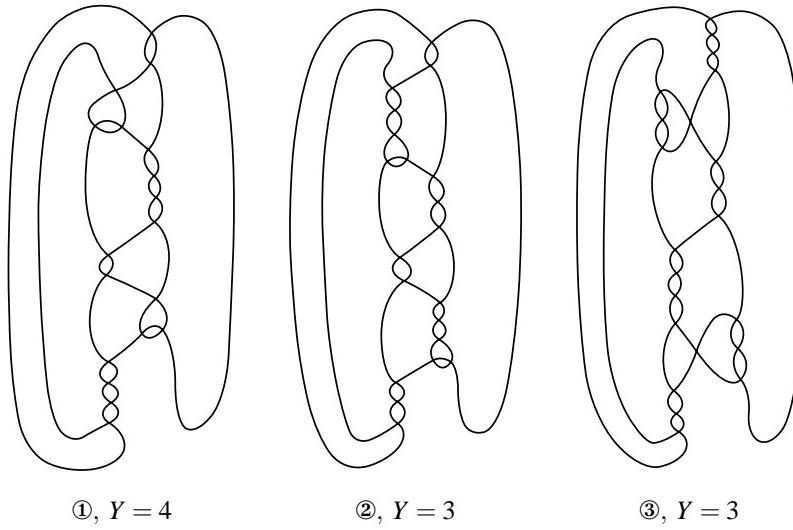
with  $T_1 = X | + Y \diagup\diagdown$ ,  $T_2 = A | + B \diagup\diagdown$  and  $T_3 = C | + D \diagup\diagdown$ . We find after multiplying out the polynomial and some manipulation

$$\langle T, T \rangle = [X(DA + BC) + Y(BD + AC)]^2.$$

Set  $X = 1$  and let  $Y = k$  vary. The rest is done by choosing small special values for  $A, B, C, D$ .

	$D$	$B$	$A$	$C$	$DA + BC$	$BD + AC$	$\sqrt{< T, T >}$	$T_2$	$T_3$
①	2	3	2	1	7	8	$7 + 8k$		
②	2	7	2	1	11	16	$11 + 16k$		
③	4	3	4	1	19	16	$19 + 16k$		

Examples of the knots thus obtained are given in figure 3 (for simplicity, just the plane curves are drawn).

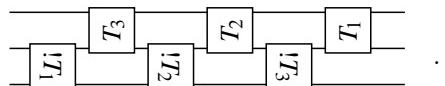


**Figure 3**

All diagrams (and hence knots [Me]) are prime for  $k \geq 1$ . Thus the only cases remaining to check are for  $\sqrt{n} = p \in \{3, 7, 11, 19\}$ . For  $p = 11$  we have  $10_{123}$ , and for  $p = 19$  we check the knots in the tables of [HT]. We obtain the examples  $12_{1019}$  (the closure of the 5-braid  $(\sigma_1 \sigma_2^{-1} \sigma_3 \sigma_4^{-1})^3$ , with  $\sigma_i$  being the Artin generators, as usual) and  $14_{18362}$  (the closure of the 3-braid  $\sigma_1^2 \sigma_2^{-3} \sigma_1^2 \sigma_2^{-2} \sigma_1^3 \sigma_2^{-2}$ ). The cases  $p = 3, 7$  were dealt with in proposition 4.1.  $\square$

**Remark 4.3** We showed that in fact we can realize any  $n$  stated in the theorem by a strongly +achiral or rational knot, so in particular by a strongly achiral knot, since an achiral rational knot is known to be strongly -achiral (see [HK]). It may be possible to exclude rational knots when allowing the further exception  $n = 25$ .

To examine the general case of  $n$  (not only perfect squares), one needs to consider larger series of examples. Because of [HK] the knots should not (only) be strongly +achiral. A natural way to modify the examples in the proof of theorem 4.4 to be -achiral is to consider (braid type) closures of tangles like



We just briefly discuss this series to explain some of the occurring difficulties.

Using the Kauffman bracket skein module coefficients

$$\begin{aligned} \boxed{T_1} &= A \overbrace{\quad} + B \circ \text{,} \\ \boxed{T_2} &= C \overbrace{\quad} + D \circ \text{,} \\ \boxed{T_3} &= X \overbrace{\quad} + Y \circ \text{,} \end{aligned}$$

one finds an expression for  $\det(K)$  as polynomial in  $A, B, C, D, X, Y$  as before, and after some manipulation arrives at  $\det(K) = f_1^2 + f_2^2$  with

$$f_1(A, B, C, D, X, Y) = X(AD + BC) + Y(AC + BD) \quad \text{and} \quad f_2(A, B, C, D, X, Y) = Y(AD - BC). \quad (12)$$

(The correctness of the square decomposition is straightforward to check, but for finding it it is helpful to notice that the substitutions  $Y = 0$  and  $A = C, B = D$  turn the knots into strongly +achiral ones, which have square determinant.)

One can conclude from (12) that no number of the form  $n = 5p^2$  with  $p \equiv 3 \pmod{4}$  prime can be written as  $f_1^2 + f_2^2$  with  $f_{1,2}$  as in (12) for  $A, B, C, D, X, Y \in \mathbb{N}$ , unless  $(A, B), (C, D)$  or  $(X, Y)$  is one of  $(0, p), (p, 0)$  or  $(p, p)$ . However, no alternating arborescent tangle has such pair of Kauffman bracket skein module coefficients. Therefore, the above series cannot realize these determinants. On the other hand, the small cases in it (for  $p \leq 11$ ) are realized by knots with Conway polyhedron 8\*.

Then one can consider more patterns and write down more complicated polynomials, each time having to show that each (at least sufficiently large) number is realized by (at least some of) these polynomials. Presently, such problems in number theory seem very difficult. (One classic example is the determination of the numbers  $G(n)$  and  $g(n)$  in Waring's problem, see for example [DHL, Ho, HW].) Therefore, conjecture 4.2 may be hard to approach as of now.

### 4.3. Determinants of unknotting number one achiral knots

We conclude our results on sums of two squares by a related, although somewhat auxiliary, consequence of the unknotting number theorem of Lickorish [Li] and its refined version given in [St3].

Let  $u_{\pm}$  denote the *signed unknotting number*, the minimal number of switches of crossings of a given sign to a crossing of the reversed sign needed to unknot a knot, or infinity if such an unknotting procedure is not available (this is somewhat different from the definition of [CL]).

Thus a knot  $K$  has  $u_+(K) = 1$  (resp.  $u_-(K) = 1$ ) if it can be unknotted by switching a positive (resp. negative) crossing in some of its diagrams.

**Proposition 4.3** Let  $K$  be a knot with  $u_+ = u_- = 1$  (for example, an achiral unknotting number one knot). Then  $\det(K)$  is the sum of two squares of coprime numbers.

**Proof.** Clearly  $\sigma(K) = 0$ , so that any of the relevant crossing changes does not alter the signature, and then by the refinement of Lickorish's theorem given in [St3], we have  $\lambda(g_{\pm}, g_{\pm}) = \pm 2/\det(K) \in \mathbb{Q}/\mathbb{Z}$  for some generators  $g_{\pm}$  of  $H_1(D_K)$ . Here  $\lambda$  is the linking form on  $H_1(D_K)$  (whose order, as discussed in the introduction, is given by  $|\Delta_K(-1)|$ ). Thus  $2l^2 = -2h^2$  for some  $l, h \in \mathbb{Z}_{\det(K)}^*$ . Then this group possesses square roots of  $-1$ .

The structure of the group  $\mathbb{Z}_n^*$  of units in  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  is known; see e.g. [Z, exercise 1, §5, p. 41]. From this structure and (5), we see that the number of square roots of  $-1$  in  $\mathbb{Z}_n^*$  (for  $n > 1$  odd) is identical to  $r_2^0(n)$ . (D. Zagier remarked to me that one can in fact give a natural bijection between the square roots of  $-1$  in  $\mathbb{Z}_n^*$  and representations of  $n$  as sum of coprime squares.)  $\square$

There are also several questions opened by proposition 4.3. Having the inclusions

$$\{\det(K) : K \text{ achiral, } u(K) = 1\} \subset \{\det(K) : u_+(K) = u_-(K) = 1\} \subset \{a^2 + b^2 : (a, b) = 1, 2 \nmid a + b\}$$

(to introduce some notation, let  $S_a$ ,  $S_{\pm}$  and  $S$  denote these three sets from left to right), the first question is whether and/or which one of these inclusions is proper (or not).

This seems much more difficult to decide than the proof of theorems 4.1 and 4.2. There is no such straightforward procedure available to exhaust all values in  $S$ , and to show a proper inclusion one will face the major problem of deciding about unknotting number one.

After a computer experiment with the knot tables and tools available to me, the smallest  $x \in S$  I could not decide to belong to  $S_{\pm}$  is 349, whereas the smallest possible  $x \in S$  with  $x \notin S_a$  is only 17. Contrarily to theorems 4.1 and 4.2, very many entries have been completed only by non-alternating knots (which have smaller determinant than the alternating ones of the same crossing number), and in fact we can use the number 17 to show that for this problem non-alternating knots definitely need to be considered.

**Example 4.2** We already quoted Crowell's result, that for a given crossing number  $n$ , the  $n$  crossing twist knot has the smallest determinant  $2n - 3$  among alternating knots  $K$  of crossing number  $n$ , except if  $n$  is odd, in which case the only knot of smaller determinant is the  $(2, n)$ -torus knot. (A more modern proof can be given for example by the Kauffman bracket, similarly to proposition 4.2.) Thus, except for the  $(2, 17)$ -torus knot, any alternating knot of determinant 17 has  $\leq 10$  crossings. A direct check shows that the only such knots of  $\sigma = 0$  are  $8_3$  and  $10_1$ . However,  $u(8_3) = 2$  as shown by Kanenobu-Murakami [KM], and that  $10_1$  cannot simultaneously have  $u_+ = u_- = 1$  follows by refining their method (see [St7]). Thus there is no alternating knot of determinant 17 with  $u_+ = u_- = 1$ , and the inclusion

$$S_{a\pm} := \{ \det(K) : K \text{ alternating}, u_+(K) = u_-(K) = 1 \} \subset S$$

is proper. (Contrarily, there is a simple non-alternating knot,  $9_{44}$ , with  $u_+ = u_- = 1$  and determinant 17.) Is it infinitely proper, i.e., is  $|S \setminus S_{a\pm}| = \infty$ ?

It is suggestive that considering the even more restricted class of rational knots, the inclusions are infinitely proper. We confirm this for achiral unknotting number one rational knots.

**Proposition 4.4**  $|S \setminus \{ \det(K) : K \text{ achiral and rational}, u(K) = 1 \}| = \infty$ . (In fact, this set contains infinitely many primes.)

**Proof.** These knots were classified in [St4, corollary 2.2] to be those with Conway notation  $(n11n)$  and  $(3(12)^n 1^4(21)^n 3)$ . It is easy to see that therefore the inclusion

$$\{ \det(K) : K \text{ achiral and rational}, u(K) = 1 \} \subset S$$

is infinitely proper. For example, the determinant of both series grows quadratically resp. exponentially in  $n$ , so that

$$\sum_{\substack{K = S(p, q) \\ \text{achiral, } u(K) = 1}} \frac{1}{p} < \infty,$$

while  $\sum_{p \in S} \frac{1}{p} = \infty$ . (By Dirichlet already  $\sum_{\substack{p \equiv 1(4) \\ \text{prime}}} \frac{1}{p} = \infty$ , see [Z, Korollar, p. 46].)  $\square$

It appears straightforward to push the method of [St4] further to show the same also for rational knots with  $u_+ = u_- = 1$  (although I have not carried out a proof in detail). By refining Kanenobu-Murakami, we first show that there is a crossing of the same sign unknotting the alternating diagram of the rational knot (see [St7]). Then consider alternating rational knot diagrams with two unknotting crossings (it does not even seem necessary to have them any more of different sign), and apply the same argument as above, using [St4, corollary 2.3] and the remark after its proof.

## 5. Enumeration of rational knots by determinant

The results of §2 can be used to enumerate rational knots by determinant. We have for example:

**Proposition 5.1** The number  $c_n$  of achiral rational knots of given determinant  $n$  is given by

$$c_n = \begin{cases} 1/2 r_2^0(n) & \text{if } n > 2 \text{ odd,} \\ 0 & \text{else.} \end{cases}$$

**Proof.** Use the fact that there is a bijective correspondence between the rational tangle  $T$  in a diagram (9) of an achiral rational knot  $K$  and its Krebes invariant  $p/q$  (with  $p \geq q$  and  $(p, q) = 1$ ) giving  $\det(K) = p^2 + q^2$ .  $\square$

From (5) we obtain then

**Corollary 5.1** The number of achiral rational knots of given determinant  $n$  is either zero or a power of two.  $\square$

As a practical application of the argument in the argument proving proposition 5.1 we can consider the achiral rational knots (1...1) and (31...13) (with the number list of even length) and the tangles  $T$  obtained from the halves of the palindromic sequence. This way one arrives to a knot theoretical explanation of the identities

$$F_{2n+1} = F_n^2 + F_{n+1}^2 \quad \text{and} \quad L_{2n+1} + 2L_{2n} = L_n^2 + L_{n+1}^2, \quad (13)$$

where  $F_n$  is the  $n$ -th Fibonacci number ( $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ ) and  $L_n$  is the  $n$ -th Lucas number ( $L_1 = 2, L_2 = 1, L_n = L_{n-1} + L_{n-2}$ ). Thus we have

**Proposition 5.2** There are achiral rational knots with determinant  $F_{2n+1}$  and  $L_{2n+1} + 2L_{2n}$  for any  $n$ , or equivalently, any (prime) number  $4x + 3$  does not divide  $F_{2n+1}$  and  $L_{2n+1} + 2L_{2n}$ .  $\square$

For  $F_{2n+1}$  this is a task I remember from an old issue of the Bulgarian journal “Matematika”. Recently I found (by electronic search) that it was conjectured in [T] and proved in [Y].

A similar enumeration can be done for arbitrary rational knots of given (odd) determinant  $n$ , and one obtains

**Proposition 5.3** The number of rational knots of determinant  $n$  ( $n > 1$  odd), counting chiral pairs once, is

$$\frac{1}{4} \left\{ \phi(n) + r_2^0(n) + 2^{\omega(n)} \right\}, \quad (14)$$

with  $r_2^0(n)$  being as in (6),  $\omega(n)$  denoting the number of different prime divisors of  $n$  and  $\phi(n)$  being Euler’s totient function.

**Proof.** We apply Burnside’s lemma on the action of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  on  $\mathbb{Z}_n^*$  given by additive inversion in the first component and multiplicative inversion in the second one. In (14), the second and third term in the braced expression come from counting the square roots of  $\mp 1$  in  $\mathbb{Z}_n^*$ . These numbers follow from the structure of this group  $\mathbb{Z}_n^*$ , as remarked in the proof of proposition 4.3.  $\square$

**Remark 5.1** The functions  $\omega(n)$  and  $\phi(n)$  are hard to calculate for sufficiently large numbers  $n$  by virtue of requiring the prime factorization of  $n$ , but the expression in terms of these classical number theoretical functions should be at least of theoretical interest.

Counting chiral pairs twice one has the somewhat simpler expression

$$\frac{1}{2} \left\{ \phi(n) + 2^{\omega(n)} \right\}.$$

In a similar way one could attempt the enumeration by  $c_p$  of unknotting number one rational knots of determinant  $p$  using [KM], seeking again an expression in terms of classical number theoretical functions. Obviously from the result of Kanenobu-Murakami we have

$$c_p \leq 2^{\omega((p+1)/2)-1} + 2^{\omega((p-1)/2)-1} - 1,$$

with the powers of two counting the representations of  $(p \pm 1)/2$  as the product of two coprime numbers  $n_{\pm}$  and  $m_{\pm}$  up to interchange of factors and the final ‘ $-1$ ’ accounting for the double representation of the twist knot for  $m_+ = m_- = 1$ . However, to obtain an exact formula, one encounters the problem that, beside the twist knot, some other knot may arise from different representations (although this does not occur often and the inequality above is very often sharp). For example, for  $p = 985$  the knot  $S(985, 288) = S(985, 697)$  occurs for the representations  $m_+ = 29$  and  $m_- = 12$ . D. Zagier informed me that he has obtained a complete description of the duplications of the Kanenobu-Murakami forms when considering  $q$  in  $S(p, q)$  only up to *additive* inversion in  $\mathbb{Z}_p^*$ . According to him, however, considering the (more relevant) *multiplicative* inversion renders the picture too complicated and number theoretically unilluminating.

## 6. Spanning trees in planar graphs and checkerboard colorings

Here we discuss an interpretation of our results of §4 in graph theoretic terms.

**Theorem 6.1** Let  $n$  be an odd natural number. Then  $n$  is the number of spanning trees in a planar self-dual graph if and only if  $n$  is the sum of two squares.

The proof of this theorem relies on the following construction linking graph and knot theory (see e.g. [Ka]). Given an alternating knot (or link) diagram  $D$ , we can associate to it its checkerboard graph.

The *checkerboard coloring* of a link diagram is a map

$$\{ \text{ regions of } D \} \rightarrow \{ \text{ black, white } \}$$

s.t. regions sharing an edge are always mapped to different colors. (A *region* is called a connected component of the complement of the plane curve of  $D$ , and an *edge* a part of the plane curve of the diagram between two crossings.)

The *checkerboard graph* of  $D$  is defined to have vertices corresponding to black regions in the checkerboard coloring of  $D$ , and an edge for each crossing  $p$  of  $D$  connecting the two black regions opposite at crossing  $p$  (so multiple edges between two vertices are allowed).

This construction defines a bijection

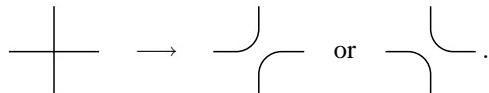
$$\{ \text{ alternating diagrams up to mirroring } \} \longleftrightarrow \{ \text{ planar graphs up to duality } \}.$$

Duality of the planar graph corresponds to switching colors in the checkerboard coloring and has the effect of mirroring the alternating diagram if we fix the sign of the crossings so that each crossing looks like  rather than . Then we have

**Lemma 6.1**  $\det(D)$  is the number of spanning trees in a checkerboard graph of  $D$  for any alternating link diagram  $D$ .

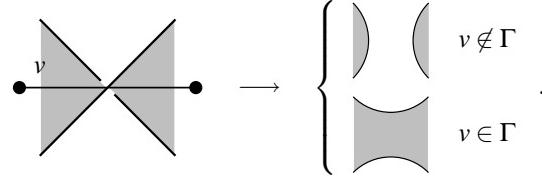
**Proof.** By the Kauffman bracket definition of the Jones polynomial  $V$ , for an alternating diagram  $D$ , the determinant  $\det(D) = |\Delta_D(-1)| = |V_D(-1)|$  can be calculated as follows (see [Kr]).

Consider  $\hat{D} \subset \mathbb{R}^2$ , the (image of) the associated immersed plane curve(s). For each crossing (self-intersection) of  $\hat{D}$  there are 2 ways to splice it:



We call a choice of splicing for each crossing a *state*. Then  $\det(D)$  is equal to the number of states so that the resulting collection of disjoint circles has only one component (a single circle). We call such states *monocyclic*.

Let  $\Gamma$  be a spanning tree of the checkerboard graph  $G$  of  $D$ . Define a state  $S(\Gamma)$  as follows: for any edge  $v$  in  $G$  set



Then  $S$  gives a bijection between monocyclic states of  $D$  and spanning trees of  $G$ .  $\square$

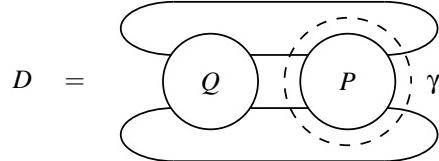
Since  $\det(K) = |\Delta_K(-1)|$  and it is known that for an  $n$ -component link  $K$ ,  $(t^{1/2} - t^{-1/2})^{n-1} \mid \Delta_K(t)$ , we have that  $2^{n-1} \mid \det(K)$ . Thus  $\det(K)$  is odd only if  $K$  is a knot. The converse is also true, since for a knot  $K$  we have  $\Delta_K(1) = 1$ , and  $\Delta_K(-1) \equiv \Delta_K(1) \pmod{2}$ .

**Proof of theorem 6.1.** If  $n$  is the number of spanning trees of a planar self-dual graph  $G$ , then its associated alternating diagram  $D$  is isotopic by  $S^2$ -moves to its mirror image. Since  $n$  is assumed odd,  $D$  is a knot diagram. Thus the number of spanning trees of  $G$ , which by the lemma is equal to  $\det(D)$ , is of the form  $a^2 + b^2$  by theorem 4.1.

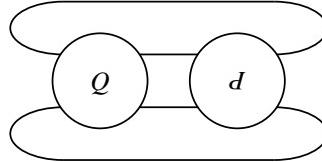
Contrarily, assume that  $n = a^2 + b^2$ . Take the checkerboard graph  $G$  of the diagram in (9) constructed in the proof of theorem 4.2. This diagram has the property of being isotopic to its mirror image by  $S^2$ -moves only (and no flypes), so that its (self-dual) checkerboard graph  $G$  is the one we sought.  $\square$

Using checkerboard colorings, we will now give a proof of proposition 4.2. We introduce first some more standard notations.

**Definition 6.1** Given a diagram  $D$  and a closed curve  $\gamma$  intersecting  $D$  in exactly four points,  $\gamma$  defines a *tangle decomposition* of  $D$ .



A *mutation* of  $D$  is obtained by removing one of the tangles in some tangle decomposition of  $D$  and replacing it by a rotated version of it by  $180^\circ$  along the axis vertical to the projection plane, or horizontal or vertical in the projection plane. For example:



(To make the orientations compatible, eventually the orientation of either  $P$  or  $Q$  must be altered.)  $\gamma$  is called the *Conway circle* for this mutation.

Note that a flype can be realized as a sequence of mutations.

There is an evident bijection between the crossings of a diagram before and after applying a mutation, so that we can trace a crossing in a sequence of mutations and identify it with its image in the transformed diagram when convenient. In particular, we can do so for a sequence of flypes.

**Proof of proposition 4.2.** We proceed by induction on  $n$ . For  $n \leq 1$  the claim is trivial.

Assume now  $L$  have  $2n$  crossings and be alternating and achiral, and  $n > 1$ . By [MT], there is a sequence of flypes (and  $S^2$ -moves) taking an alternating diagram  $D$  of  $L$  into its mirror image  $!D$ .

Fix a crossing  $p$  in  $D$  and let  $p'$  be the crossing in  $D$  whose trace under the flypes taking  $D$  to  $!D$  takes it to the mirror image of  $p$  in  $!D$ .

Since the only diagram in which both splicings of a crossing give a nugatory crossing is the Hopf link diagram, for each  $p$  and  $p'$  there is a splicing not producing a nugatory crossing. We call such splicing a *non-nugatory splicing*, otherwise call the splicing *nugatory*.

We can distinguish the two splicings at  $p$  according to the colors of the regions in the checkerboard coloring they join. We thus call the splicings *black* or *white*.

Since whether the black or white splicing gives a nugatory crossing is invariant under fypes, the choice of non-nugatory splicing at  $p$  and  $p'$  between black or white splicing can be made to be the opposite. Call  $D_p$  (resp.  $D_{p'}$ ) the diagrams obtained from  $D$  after the so chosen splicing at  $p$  (resp.  $p'$ ), and  $D'$  the diagram resulting after performing both splicings. Since  $D_p$  and  $D_{p'}$  have no nugatory crossings,  $D'$  is non-split.

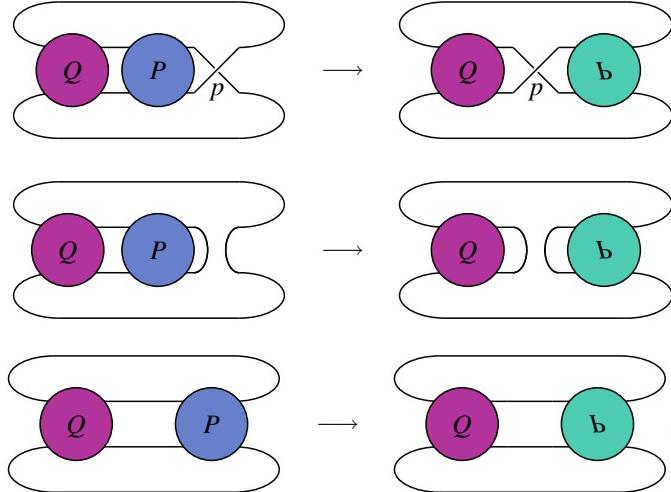
We claim that  $D'$  has no nugatory crossings. To see this, use that  $p$  and  $p'$  join two pairs of regions of opposite color in the checkerboard coloring of  $D$ . If  $D'$  has a nugatory crossing  $q$ , then there would be a closed plane curve  $\gamma$  intersecting  $D'$  (transversely) only in  $q$ , and lying in some (without loss of generality) white region of the checkerboard coloring of  $D'$ .



But then  $\gamma$  would persist by undoing the splicing joining the two black regions, and thus  $q$  would be nugatory in one of  $D_p$  or  $D_{p'}$ , too, a contradiction.

Since we need the plane curve argument later, let us for convenience call a curve  $\gamma$  through black (resp. white) regions of  $D$  intersecting  $D$  in crossings  $c_1, \dots, c_n$  a *black* (resp. *white*) *curve* through  $c_1, \dots, c_n$ .

We also claim that  $D'$  depicts the mutant of an achiral link. This follows, because the fypes carrying  $D$  to  $!D$  also carry  $D'$  to  $!D'$ , modulo mutation. To see this, the only problematic case to consider is when a fype at  $p$  must be performed. Then



which are both mutations. (Altering the way of building connected sums out of the prime factor diagrams is also considered a mutation.)

Since mutation does not alter the determinant, we have by induction  $\det(D') \geq (n-1)(n-4)$ .

Now let  $D'_b$  and  $D'_w$  be the diagrams obtained from  $D$  when at both  $p$  and  $p'$  the black resp. white splicings are applied, and  $L_b$  and  $L_w$  the alternating links they represent.

When a nugatory white splicing at a crossing  $r$  in  $D$  renders a crossing  $q$  nugatory, then we have a white curve  $\gamma$  as in (15) though  $p$  and  $q$ . Similarly if two white splicings at crossings  $r$  and  $s$  render  $q$  nugatory, then we have a white curve  $\gamma$  in  $D$  through  $q, r$  and  $s$ .

Assume now, a crossing  $q \notin \{p, p'\}$  is nugatory in both  $D'_b$  and  $D'_w$ . Then we have a white curve  $\gamma_{q,w}$  and a black curve  $\gamma_{q,b}$  in  $D$  through  $q$  and at least one of  $p$  and  $p'$  (and no other crossing different from  $p$  and  $p'$ ). By the Jordan curve theorem,  $|\gamma_{q,w} \cap \gamma_{q,b}| = 2$ .

If  $\gamma_{q,b} \cap D = \gamma_{q,w} \cap D$  and  $|\gamma_{q,w} \cap D| = 2$ , then  $D$  is a Hopf link diagram, which we excluded.

Thus  $\{|\gamma_{q,b} \cap D|, |\gamma_{q,w} \cap D|\} = \{2, 3\}$ . We claim that for each of the two choices  $|\gamma_{q,b} \cap D| = 3$  and  $|\gamma_{q,w} \cap D| = 3$  there is at most one  $q \notin \{p, p'\}$  satisfying these conditions. This is so, because  $\gamma_{q,b} \cap D = \{q, p, p'\}$  and  $\gamma_{q',b} \cap D = \{q', p, p'\}$  imply that  $q$  and  $q'$  have the same pair of opposite black regions. (These are the regions adjacent to exactly one of  $p$  and  $p'$ .) Then every white curve through  $q'$  must pass through  $q$ , contradicting  $\gamma_{q',w} \cap D \subset \{q', p, p'\}$  if  $q \neq q'$ . (For  $|\gamma_{q,w} \cap D| = 3$  switch black and white.)

Therefore, each  $q \notin \{p, p'\}$  is not nugatory at least in one of  $D'_b$  and  $D'_w$ , with at most 2 exceptions. Hence,  $c(L_b) + c(L_w) \geq 2n - 4$ , and by Crowell's result  $\det(D'_b) + \det(D'_w) \geq 2n - 4$ .

Thus finally

$$\det(D) \geq \det(D') + \det(D'_b) + \det(D'_w) \geq (n-1)(n-4) + 2n - 4 = n(n-3). \quad \square$$

As a graph theoretical application, we obtain:

**Corollary 6.1** The number of spanning trees in a planar connected self-dual graph with  $2n$  edges is at least  $n(n-3)$ .

$\square$

One can also reformulate theorem 4.4 graph-theoretically. For this one remarks that by [Me] an alternating diagram of a composite alternating knot is composite, and checkerboard graphs of such diagrams have a *cut vertex*. (This is a vertex, which when removed together with all its incident edges, disconnects the graph.) With the same argument as in the proof of theorem 6.1 then one has:

**Theorem 6.2** Let  $n$  be an odd perfect square. Then  $n$  is the number of spanning trees in a planar self-dual graph without cut vertex if and only if  $n \neq 1, 9, 49$ .  $\square$

In the same way one can pose a conjecture which is slightly stronger than conjecture 4.2:

**Conjecture 6.1** Let  $n$  be an odd natural number. Then  $n$  is the number of spanning trees in a planar self-dual graph without cut vertex if and only if  $n$  is the sum of two squares and  $n \neq 1, 9, 49$ .

## 7. The leading coefficients of the Alexander and HOMFLY polynomial

In this final section we explain another, apparently unrelated, but also very striking occurrence of squares in connection with invariants of achiral knots, namely in the leading coefficients of their Alexander polynomial, and discuss a possible generalization of this property to the HOMFLY, or skein, polynomial. Our results have been obtained independently (but later) by C. Weber and Q. H. Cân VÂN [VW].

The skein polynomial  $P$  is a Laurent polynomial in two variables  $l$  and  $m$  of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1}P(\text{\diagup\diagdown}) + lP(\text{\diagdown\diagup}) = -mP(\text{\diagup})(\text{\diagdown}). \quad (16)$$

This convention uses the variables of [LM], but differs from theirs by the interchange of  $l$  and  $l^{-1}$ .

There is a classic substitution formula (see [LM]), expressing the Alexander polynomial  $\Delta$ , for the normalization so that  $\Delta(t) = \Delta(1/t)$  and  $\Delta(1) = 1$ , as a special case of the HOMFLY polynomial:

$$\Delta(t) = P(i, i(t^{1/2} - t^{-1/2})). \quad (17)$$

We will denote by  $\max \deg_m P$  the *maximal degree* of  $m$  in  $P$ , and by  $\max \operatorname{cf}_m P$  the *leading coefficient* of  $m$  in  $P$ . Similarly we will write  $\max \deg \Delta$  and  $\max \operatorname{cf} \Delta$ .

## 7.1. Problems and partial solutions

If  $K$  is an alternating knot, then the HOMFLY polynomial  $P_K \in \mathbb{Z}[m^2, l^{\pm 2}]$  is known to be of the form

$$a_{2g}(l)m^{2g} + (\text{lower } m\text{-degree terms}),$$

with  $a_{2g} \in \mathbb{Z}[l^2, l^{-2}]$  being a non-zero Laurent polynomial in  $l^2$  and  $g = g(K)$  the *genus* of  $K$ , the minimal genus of an embedded oriented surface  $S \subset \mathbb{R}^3$  with  $\partial S = K$ . (See [Cr].) If  $K$  is achiral, then  $a_{2g}(l^{-2}) = a_{2g}(l^2)$ , that is,  $a_{2g}$  (and, in fact, all the other coefficients of  $m$  in  $P_K$ ) is self-conjugate.

The main problems we consider here can be formulated as follows.

**Question 7.1** Is  $\max \text{cf} \Delta_K$  for an achiral knot  $K$  always a square up to sign, and if  $\Delta$  is normalized so that  $\Delta(t) = \Delta(1/t)$  and  $\Delta(1) = 1$ , is the sign  $\text{sgn}(\max \text{cf} \Delta_K)$  always given by  $(-1)^{\max \deg \Delta_K}$ ?

The questions on  $\Delta$  can be generalized to  $P$ .

**Question 7.2** For which large knot classes is it true that achiral knots have  $\max \text{cf}_m P$  of the form  $f(l^2)f(l^{-2})$  for some  $f \in \mathbb{Z}[l]$ ?

This is true for several special cases. It appears convenient to compile them into one single statement.

Recall, that a knot  $K$  is *fibered*, if  $S^3 \setminus K$  fibers over  $S^1$  (with fiber being a minimal genus Seifert surface for  $K$ ), and *homogeneous*, if it has a diagram  $D$  containing in each connected component of the complement (in  $\mathbb{R}^2$ ) of the Seifert circles of  $D$  (called *block* in [Cr, §1]) only crossings of the same sign.

**Proposition 7.1** Let  $K$  be an  $(+/-)$ achiral knot. Then  $\max \text{cf}_m P_K$  is of the form  $f(l^2)f(l^{-2})$  for some  $f \in \mathbb{Z}[x]$ , if

- 1)  $K$  is a fibered homogeneous knot,
- 2)  $K$  is a homogeneous knot of crossing number at most 16, or
- 3)  $K$  is an alternating knot.

From formula (17) it is straightforward that whenever the leading  $m$ -coefficient of  $P_K$  is of the above form, both the modulus and sign of  $\max \text{cf} \Delta_K$  are as requested in question 7.1. For these properties we have some more situations where they can be established. We give again the so far complete list of such cases, even if some of them are trivial.

**Proposition 7.2** Let  $K$  be an achiral knot. Then  $|\max \text{cf} \Delta_K|$  is a square, if

- 1)  $K$  is a fibered knot,
- 2)  $K$  is a knot of crossing number at most 16,
- 3)  $K$  is an alternating knot,
- 4)  $K$  is strongly achiral, or
- 5)  $K$  is negative achiral.

Moreover,  $\text{sgn}(\max \text{cf} \Delta_K) = (-1)^{\max \deg \Delta_K}$ , if

- 6)  $K$  is a fibered homogeneous knot,
- 7)  $K$  is a knot of crossing number at most 16,
- 8)  $K$  is an alternating knot,
- 9)  $K$  is strongly achiral, or

10)  $K$  is negative achiral.

**Remark 7.1** We omitted to explicitly mention rational knots in proposition 7.2, as we already remarked that they are alternating, and the achiral ones are strongly  $(-)$ achiral.

In the following subsections we will collect the arguments establishing the conjectured properties in the indicated special cases. Some of them are well-known, or a matter of electronic verification, and thus do not deserve separate proof. These parts are briefly discussed first. Our main result are the statements in the alternating case, which are proved subsequently.

## 7.2. Some known and experimental results

Question 7.1 was the (chronologically) first question I came across, addressing special properties of the leading coefficients of the Alexander and skein polynomial of achiral knots.

This question came up when considering the formula

$$\max \text{cf} \Delta_K = \pm 2^{-2g} \prod_{i=1}^{2g} a_i$$

for a rational knot  $K = (a_1 \dots a_{2g})$  with all  $a_i \neq 0$  even. If  $K$  is achiral, the sequence  $(a_1, \dots, a_{2g})$  is palindromic, and so we have, up to the sign, the requested property for rational knots. A further larger class of achiral knots satisfying the conjectured condition are the strongly achiral knots (see the Theorem of [HK]). Subsequently, I verified all (prime) knots in Thistlethwaite's tables [HT] up to 16 crossings (note, that the property for a composite knot will follow from that of its factors), and found no counterexample.

Although there seems much evidence for a positive answer to question 7.2, its diagrammatic, and not topological, origin (see §7.4) suggests that it may not be true in general, but at least on some (diagrammatically defined) nice knot classes, for example alternating knots.

We now collect the arguments that prove the easier cases of propositions 7.1 and 7.2.

### Proof of proposition 7.1 except part 3) and proposition 7.2 except part 3).

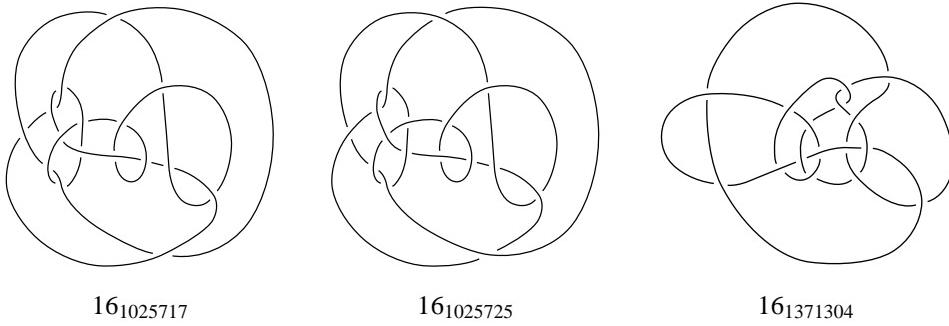
*alternating knots.* As said, we will deal with the squareness and HOMFLY polynomial later. As for the further-going question on the sign, the positive answer follows from the alternation of the coefficients of  $\Delta$  proved by Crowell in [C], and the property  $\Delta(-1) > 0$  following from Murasugi's trick, as explained in §2.

*knots with at most 16 crossings.* The answer is also 'yes' for  $\leq 16$  crossing knots. This follows from some experimental results related to question 7.2. It is clear that an answer 'yes' to question 7.2 implies the same answer to question 7.1. This time a computer experiment found that the answer is not positive in general, but the examples showing exceptional behaviour are not quite simple, and required to use the full extent of the tables presently available. Among  $\leq 16$  crossing knots, only three fail to have this property: 16<sub>1025717</sub>, 16<sub>1025725</sub> and 16<sub>1371304</sub>. They are all +achiral and have  $P = m^8(l^{-2} + 3 + l^2) + O(m^6)$ . See figure 4. Since the Alexander polynomial of the three knots has degree 4 and leading coefficient 1, they still conform to the properties requested in question 7.1. Also, at least the first two knots in figure 4 were found to be fibered by the method of [Ga], showing that the homogeneity assumption is essential in part 1) of proposition 7.1.

*fibered knots.* On the other hand, a class of knots where the squareness property of  $\max \text{cf} \Delta$  is trivial, are the fibered knots, since then  $\max \text{cf} \Delta = \pm 1$ . For fibered homogeneous (in particular, alternating) knots, the other properties also follow easily from known results, because by [Cr, corollaries 4.3 and 5.3] and [MP] we have for such knots that  $\max \text{cf}_m P = l^k$  for some  $k \in 2\mathbb{Z}$ , and then achirality shows  $k = 0$ .

*strongly achiral knots.* For strongly achiral knots the claims of proposition 7.2 follow directly from the results of [HK] stated in theorem 4.5. The only non-obvious property may be the sign of  $\max \text{cf} \Delta$  for a strongly negative amphicheiral knot. To see this, first normalize the polynomial  $F$  found by the theorem by some  $+t^n$  so that  $\min \deg F = -\max \deg F$ . Then we have

$$F(t) = \pm F(-t^{-1}). \quad (18)$$



**Figure 4:** Three +achiral knots with  $\max \text{cf}_m P$  not of the form  $f(l^2)f(l^{-2})$ .

If we normalize  $\Delta$  so that  $\min \deg \Delta = -\max \deg \Delta$  and  $\Delta(1) = 1$ , then the minimal and maximal degrees show that we must have  $n = 0$  in

$$\Delta(t^2) = \pm t^n F(t)F(t^{-1}),$$

and the value at  $t = 1$  shows that we must have the positive sign. Denote by  $[X]_i$  the coefficient of  $t^i$  in  $X \in \mathcal{L}$ . Then, since  $\Delta(1) = 1$  and  $\Delta(t) = \Delta(t^{-1})$ , the absolute term  $[\Delta(t)]_0$  of  $\Delta(t)$  is odd. Thus the same is true for  $\Delta(t^2) = F(t)F(t^{-1})$ . But

$$[F(t)F(t^{-1})]_0 = \sum_{i=\min \deg F}^{\max \deg F} [F(t)]_i^2,$$

and so from (18) we conclude that  $F$  must have non-zero absolute term. This determines the sign in (18) to be positive, and then  $\max \text{cf} F = \pm \min \text{cf} F$  dependingly on the parity of  $\max \deg F = \max \deg \Delta$ .

*negative amphicheiral knots.* Hartley [Ha] has extended the result of [HK] for strongly negative amphicheiral knots to arbitrary negative amphicheiral knots. Thus the claim follows from the previous argument.  $\square$

**Remark 7.2** In fact, in [Kw], Kawauchi conjectures that the property of the Alexander polynomial of a strongly negative amphicheiral knot he proves with Hartley in [HK], and later Hartley [Ha] generalizes to an arbitrary negative amphicheiral knot, extends to the Alexander polynomial of an arbitrary amphicheiral knot. This conjecture clearly implies a positive answer to question 7.1. Kawauchi's conjecture is true in particular for 2-bridge knots, since in [HK] he shows that all amphicheiral 2-bridge knots are strongly negative amphicheiral. I verified the conjecture for all prime amphicheiral knots of  $\leq 16$  crossings. Note that Hartley also obtains a condition for positive amphicheiral knots, but it is too weak to address any of our questions.

**Remark 7.3** Fibered homogeneous knots contain the homogeneous braid knots of [S], but also many more. For example, there are 15 fibered homogeneous prime 10 crossing knots, among them 12 alternating and 2 positive ones, which can be shown by the work of [Cr] and an easy computer check not to have homogeneous braid representations:  $10_{60}, 10_{69}, 10_{73}, 10_{75}, 10_{78}, 10_{81}, 10_{89}, 10_{96}, 10_{105}, 10_{107}, 10_{110}, 10_{115}, 10_{154}, 10_{156}$  and  $10_{161}$ .

### 7.3. Two general statements

The remaining cases of propositions 7.2 and 7.1 are included in two more general theorems. The first one generalizes the result of [MP2], where it was shown that for an alternating amphicheiral knot the leading coefficient of the Alexander polynomial is not a prime.

**Theorem 7.1** Let a knot  $K$  have a homogeneous diagram  $D$  which can be turned into its mirror image (possibly with opposite orientation) by a sequence of mutations and  $S^2$ -moves (changes of the unbounded region). Then  $|\max \text{cf} \Delta_K|$  is a square.

**Proof of theorem 7.1.** Let  $\tilde{D}$  denote the mutation equivalence class of a knot diagram  $D$  (that is, the set of all diagrams that can be obtained from  $D$  by a sequence of mutations). Assume  $D$  to be non-split (the split case easily reduces to the non-split one). We consider orientation reversal as a special type of mutation, so a diagram and its inverse belong to the same mutation equivalence class.

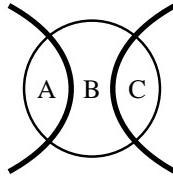
As in [Cr, §1], the Seifert picture of  $D$  defines a decomposition of  $D$  into the *\*-product (or Murasugi sum)* of special alternating diagrams  $D_1, \dots, D_n$ , called *blocks*. These diagrams may not be prime. Let  $D_{i,1}, \dots, D_{i,n_i}$  be the prime components of  $D_i$ . Note that all  $D_{i,j}$  are positive or negative (depending on  $D_i$ ). They will have no nugatory crossings if  $D$  has neither.

Define

$$I(D) := \{\tilde{D}_{i,j}\}_{i=1,\dots,n, j=1,\dots,n_i}.$$

Here a set is to be understood with the order of its elements ignored, but with their multiplicity counted (i.e.,  $\{1, 1, 2, 3\} = \{1, 1, 3, 2\} \neq \{1, 2, 3\}$ ).

Now apply a mutation on  $D$ . The Seifert picture separates the Conway circle into 3 parts  $A$ ,  $B$  and  $C$ .



Because the Conway circle intersects the Seifert picture only in 4 points, all parts  $A$ ,  $B$  and  $C$  represent connected components of the blocks in  $D$  they belong to (or possibly connected sums of several such connected components).

Mutation then has the effect of applying mutation on  $B$  and interchanging and/or reversing  $A$  and  $C$ . Therefore,  $I(D) = I(D')$  for any iterated mutant diagram  $D'$  of  $D$ .

If  $D$  has the property assumed in the theorem, then  $I(D) = I(\overline{D})$ , or

$$\{\tilde{D}_{i,j}\}_{i=1,\dots,n, j=1,\dots,n_i} = \{\widetilde{!D_{i,j}}\}_{i=1,\dots,n, j=1,\dots,n_i}.$$

Let  $\phi : \{\tilde{D}_{i,j}\} \rightarrow \{\tilde{D}_{i,j}\}$  be the bijection induced by  $\tilde{D}_{i,j} \mapsto \widetilde{!D_{i,j}}$ .

Since mutation preserves the writhe,  $\phi$  has no fixpoints (unless some  $D_{i,j}$  has no crossings, in which case  $D$  is split). Thus  $\phi$  descends to a bijection

$$\phi : \{\tilde{D}_{i,j} : D_{i,j} \text{ positive}\} \rightarrow \{\tilde{D}_{i,j} : D_{i,j} \text{ negative}\}.$$

Then by [Mu2],  $\max \text{cf} \Delta$  is multiplicative under *\*-product*, and hence

$$\begin{aligned} \max \text{cf} \Delta_D &= \prod_{i,j} \max \text{cf} \Delta_{D_{i,j}} \\ &= \prod_{i,j : D_{i,j} \text{ positive}} \max \text{cf} \Delta_{D_{i,j}} \cdot \max \text{cf} \Delta_{!D_{i,j}} \\ &= \prod_{i,j : D_{i,j} \text{ positive}} \max \text{cf} \Delta_{D_{i,j}} \cdot \pm \max \text{cf} \Delta_{D_{i,j}} \\ &= \pm \left( \prod_{i,j : D_{i,j} \text{ positive}} \max \text{cf} \Delta_{D_{i,j}} \right)^2, \end{aligned}$$

as desired.  $\square$

**Theorem 7.2** Under the same assumption as theorem 7.1 we have  $\max \text{cf}_m P(D) = f(l^2)f(l^{-2})$  for some  $f \in \mathbb{Z}[x]$ .

**Proof.** Using [MP2] instead of [Mu2], we obtain  $\max \text{cf}_m P(D) = f(l)f(l^{-1})$ . Since  $\max \text{cf}_m P(D)$  has only even powers of  $l$ , the result follows.  $\square$

**Corollary 7.1** For any alternating achiral knot  $K$ ,  $|\max \text{cf} \Delta_K|$  is a square and  $\max \text{cf}_m P_K = f(l^2)f(l^{-2})$ .

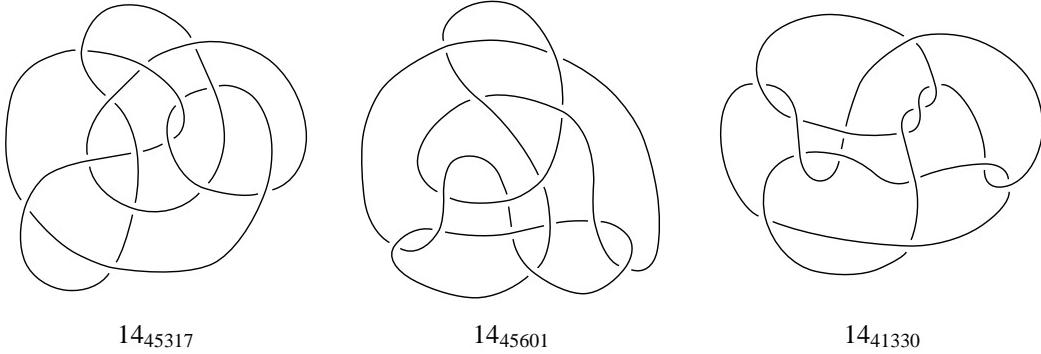
**Proof.** Use that an alternating diagram is homogeneous, [MT], and that a flype can be realized as a sequence of mutations.  $\square$

Note, that in the case of  $\Delta$  we obtain an exact condition when a number occurs as  $|\max \text{cf} \Delta_K|$  for an alternating achiral knot  $K$  (since the other implication is trivial).

**Remark 7.4** We need the homogeneity of  $D$  only to assure that all  $D_{i,j}$  are positive or negative. (This weaker property is invariant under mutations, whereas homogeneity is not.) Thus the theorems could be formulated even slightly more generally, but then also more technically.

Although the corollary is the most interesting special case of the theorems, they give indeed more general statements.

**Example 7.1** The non-alternating achiral knots 14<sub>45317</sub> and 14<sub>45601</sub> have unique minimal diagrams (which therefore must be transformable into their mirror images by  $S^2$ -moves only), which are both homogeneous (of genus 5 and 4, respectively).



**Figure 5**

**Remark 7.5** Here we consider (and mean unique) minimal diagrams only up to  $S^2$ -moves, and not as in [HT] up to  $S^2$ -moves and mirroring. There are achiral knots with a unique minimal diagram up to  $S^2$ -moves and mirroring, but corresponding to two different (mirrored) diagrams up to  $S^2$ -moves only, which are not interconvertible by flypes. One such example is 14<sub>41330</sub>.

**Remark 7.6** If a knot has a diagram  $D$ , which can be transformed into its (possibly reverted) obverse by moves in  $S^2$  and flypes, then it also has a diagram  $D'$ , which can be transformed into its obverse by moves in  $S^2$  only. This follows from the fact that mirroring and moves in  $S^2$  take the flyping circuits of  $D$  into each other (for the definition of flyping circuits see [ST, §3]), and flyping in a flyping circuit is independent from the other ones.  $D'$  can be obtained by appropriate flypes from  $D$ . (For an analogous statement about the checkerboard graphs, see [DH].)

## 7.4. Some diagrammatic questions

The two theorems of §7.3 suggest the diagrammatic arguments motivating questions 7.1 and 7.2. We conclude with a more detailed problem concerning possible generalizations. In order to make the result of [MP] work, we need to consider  $P$ -maximal diagrams.

**Definition 7.1** Call a link diagram  $D$  with  $c(D)$  crossings and  $s(D)$  Seifert circles  $P$ -maximal, if  $\max \deg_m P(D) = c(D) - s(D) + 1$ .

In [Mo], Morton showed that the one inequality  $\max \deg_m P(D) \leq c(D) - s(D) + 1$  holds for any arbitrary link diagram, and used this to show that there are knots  $K$ , which do not possess a diagram  $D$  with  $g(D) = g(K)$  (a fact that also implicitly follows from [Wh]). Here  $g$  denotes the *genus* of a knot or knot diagram, for latter being defined by  $g(D) = 1/2(c(D) - s(D) + 1)$ , which is the genus of the canonical Seifert surface associated to  $D$  (see [Ad, §4.3] or [Ro]).

In [Cr] it was shown that homogeneous diagrams are  $P$ -maximal. Many knots have  $P$ -maximal diagrams – beside the homogeneous knots, for example all (other) knots in Rolfsen's tables [Ro, appendix] and also all the 11 and 12 crossing knots tabulated in [HT]. However, some knots do not – in [St2, fig. 9] we gave four examples of 15 crossings.

In [MP] it was shown that  $\max \text{cf}_m P$  is multiplicative under  $*$ -product of  $P$ -maximal diagrams. This is the link between the above polynomial conjectures and the diagrammatic problems, which we summarize in the question below.

**Question 7.3** Does any  $(+/-)$ achiral knot (or an achiral knot in which large knot class) have a diagram that can be

- A) transformed into its (possibly reverted) obverse by moves in  $S^2$  (changes of the unbounded region), or
- B) represented as the iterated connected and Murasugi sum of (some even number of)  $P$ -maximal link diagrams  $D_i$  with  $\{D_i\}$  being mutually obverse (up to orientation and  $S^2$ -moves) in pairs?

As a motivation, we mention briefly some related partial cases and implications.

Remarks on part A):

- By considering the blocks of such a diagram, we see that it is the  $*$ -product of special diagrams  $D_i$ , such that each  $D_i$  is transformable by  $S^2$ -moves into the obverse of itself, or of some other  $D_j$ ,  $j \neq i$ .
- If the answer is ‘yes’ for some *homogeneous* diagram (which in particular happens by [MT] for alternating diagrams not admitting flypes), no blocks transform onto their own obverses, and [MP] shows a positive answer to question 7.2.

The  $(l-)$ coefficients of  $\max \text{cf}_m P$  of the three knots in figure 4 do not alternate in sign, and so these knots cannot be homogeneous (beware of the different convention for  $P$  in [Cr]!). Still, the first two knots have a  $P$ -maximal diagram of the type requested in part A). It can be obtained from the one given in the figure by a flype. Thus theorem 7.2 does not hold under the weaker assumption the diagram to be  $P$ -maximal instead of homogeneous.

- The answer is ‘yes’ for rational knots. The (palindromic) expression  $(a_1 \dots a_n a_n \dots a_1)$  with all  $2g$  even numbers  $a_i \neq 0$  gives a rational diagram (of the form  $D * !D$ , where  $D$  is a connected sum of diagrams of reversely oriented  $(2, a_{2i})$ -torus links), having the desired property.
- By remark 7.6, one can equivalently also allow moves in  $S^2$  and flypes, so the answer is in particular positive for alternating knots by [MT].

Remarks on part B):

- By [MP], a positive answer implies a positive answer also for question 7.2 (thus in particular the answer is negative for the knots on figure 4). This is the motivation for proposing question 7.2 after arriving to question 7.1.
- In turn, a positive answer to B) is implied by a positive answer to A) for homogeneous diagrams. This is the motivation for proposing part B).

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